Int. J. Solids Structures Vol. 23, No. 6, pp. 797–817, 1987
Printed in Great Britain.

0020-7683/87 $3.00+.00

ELASTIC ANALYSES OF CIRCULAR CYLINDRICAL SHELLS BY ROD THEORY INCLUDING DISTORTION OF CROSS-SECTION

HIDEO TAKABATAKE
Kanazawa Institute of Technology, 7-1, Ogigaoka Nonoichimachi, Isikawa 921, Japan

and

OSAMU MATSUOKA
Nagoya University, Nagoya 464, Japan

(*Received* 24 *June* 1985; *in revised form* 15 *June* 1986)

Abstract—A set of governing equations for the linear theory of a circular cylindrical shell such as tanks and silos is presented explicitly from rod theory including the distortion of the transverse cross-section. It is assumed that the deformation in the rod consists of a fundamental deformation, which can be expressed by displacements and rotations of the axial line of the rod, and a higher-ordered deformation caused by the warping and distortion of the transverse cross-section. Also, it is assumed that the higher-ordered deformation adds to the fundamental deformation. The higher-ordered deformation is expressed by the circumferential Fourier series expansion. Then, the simplifications of the obtained governing equations are carried out by using the classical hypotheses in rods. Finally, to examine the derived theory, a static problem of a cantilevered circular cylindrical shell has been solved analytically by the Bernoulli–Euler beam theory, by the Timoshenko beam theory, and by a theory including distortion. From the numerical results, it is concluded that the distortion is large compared to the transverse deflection of the axial line of rods and that the influence of the distortion on the transverse deflection of the axial line of rods is negligible.

1. INTRODUCTION

For analyses of cantilevered circular cylindrical shells used as tanks and silos, there are two theories, shell theory and rod theory.

Shell theory is generally applied to shells, but it has the following disadvantages.

(1) Since the accuracy of the solution is influenced considerably by boundary conditions, the exact boundary conditions need to be established. However, they are difficult to establish in practice.

(2) For particular loads, simple solutions are useful, for example, Seide[1] presented a simplified solution to an end-loaded cantilevered cylinder with a rigid end-ring. However, for general loads the simplified solutions are not available and the analysis has to be based on numerical calculations by computer, and is therefore very expensive. Hence, it is often impossible to use shell theory in the preliminary design of shells.

On the other hand, rod theory is applied to grasp the behavior of cylindrical shells in broader aspects. Rod theory has the following advantages.

(1) Because of the macroscopic treatment, the governing equations and boundary conditions can be expressed in simpler forms than those from shell theory, hence the analysis is easy.

(2) The mechanical behavior can be macroscopically obtained.

However, rod theory has a fault, the distortion of the transverse cross-section of the shells is not considered. To cover up this fault and to improve the accuracy to the same degree as shell theory, it is necessary to establish the highly accurate rod theory including the distortion of the transverse cross-section. Recently, Hangai and Choi[2] presented the local buckling analysis of cantilevered cylindrical shells subjected to a transverse end-shearing force, under constraining the distortion of the cross-section at both ends. However, rod theory including the distortion of the transverse cross-section to cylindrical shells, has not been established for general loads and general boundary conditions.

Meanwhile, the general theory of a Cosserat curve which models a rod-like body with only two directors includes distortion of the transverse cross-section since the directors are allowed to change in length. However, this simple theory cannot express practically the higher-ordered distortion by two directors. Moreover, this simple theory does not include warping since plane cross-sections remain plane.

The aim of this paper is to formulate the general governing equations of motion to cylindrical shells by rod theory including the distortion of the cross-section.

The formulation of the governing equations is presented for infinitesimal deformations through the principle of virtual work, under the assumption that the deformation of the cylindrical shell consists of the fundamental deformation, which can be expressed by displacements and rotations of the axial line of the rod, and the higher-ordered deformation, which is produced by the warping and distortion of the cross-section. Subsequently, for practical use of the derived theory, the number of unknowns is decreased by means of classical hypotheses in rod theory. Finally, in order to examine the proposed theory, analytical solutions for the Bernoulli–Euler beam, for the Timoshenko beam, and for the theory including only the distortion of the transverse cross-section, are presented to a static problem where a uniform load acts on a cantilevered cylindrical shell. Then the effect of the distortion of the transverse cross-section on the transverse displacement of the axial line of the rod is discussed.

Meanwhile, in the dynamic problem of liquid storage tanks, there is a problem of a coupling effect between the liquid and the shell. Recently, Haroun[3] showed that this coupling effect is negligible. Consequently, in practice, the dynamic problem of liquid storage cylindrical shells can be treated as a problem where dynamic pressures, given by either the Housner theory[4] or the velocity potential theory for rigid shells, act on the empty tank, namely, the shell. Since this treatment, neglecting the coupling effect, simplifies remarkably the analysis of cylindrical shells, this paper deals with the dynamic pressures as usual loads acting on the rod.

As for tensor notations, Latin indices take the values of 1, 2, and 3; whereas Greek indices take the values of 1 and 2. On these notations, Latin indices i, j, and k are used mainly in the x-coordinate system and Latin indices a, b, and c mainly in the y-coordinate system. On the other hand, Greek indices α, β, and λ are used mainly in the x-coordinate system, and Greek indices ξ, η, and ζ mainly in the y-coordinate system.

2. ASSUMPTIONS

In the analysis the following assumptions are employed.

(1) Deformations are infinitesimal.

(2) The stress–strain relations are linear.

(3) A circular cylindrical shell is composed of an isotropic and elastic material and is uniform and thin walled.

(4) Initial imperfections are negligible.

3. DEFORMATIONS OF RODS

Let us consider a uniform thin-walled circular cylindrical shell embedded in a Euclidian 3-space. The shell does not hold initial imperfections. The shape of a shell is adequately defined by describing the geometry of its middle surface, which is a surface that bisects the shell thickness t at each point. Now, two sets of coordinate systems x and y are prepared, as shown in Fig. 1.

In the coordinate system x^i $(i = 1, 2, 3)$, the coordinate axis x^3 takes the axial curve that is the centroidal axis, and the transverse axes x^α $(\alpha = 1, 2)$ take the directions of the base vectors $\mathbf{A}_\alpha$, respectively, in the transverse cross-section prescribed by the value of x^3. Herein values of x^α always indicate a point located on the middle surface of the shell.

On the other hand, the y^a-coordinate system $(a = 1, 2, 3)$ is a curvilinear coordinate system defined on the middle line in the transverse cross-section prescribed by the values

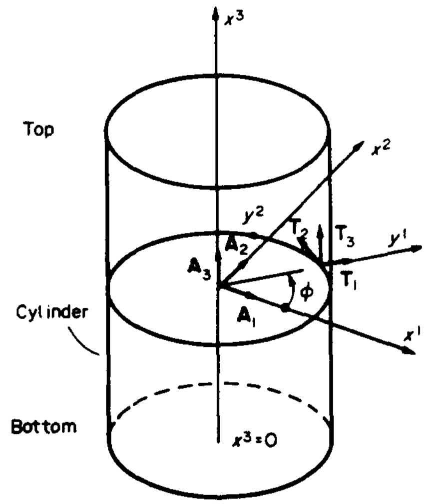

Fig. 1. Coordinates and base vectors.

of x^3. The y^1-coordinate axis takes the direction of the thickness of the shell prescribed by the base vector $\mathbf{T}_1$; and the y^2-coordinate axis takes the direction of the middle line possessing the base vector $\mathbf{T}_2$ for the tangential vector; and the y^3-coordinate axis prescribed by the base vector $\mathbf{T}_3$ agrees with the x^3-axis.

The x^i-coordinate system is used in the macroscopic expression for rods, meanwhile the y^a-coordinate system is used in the microscopic expression, such as indicating warping and the distortion of the transverse cross-section. Thus, it is convenient from the engineering point of view to use two coordinate systems properly.

The position vector $\mathbf{R}$ of a point on the middle surface in the undeformed cylindrical shell is a function of an angle ϕ and the x^3-axis with $x^\alpha = x^\alpha(\phi)$. Here an angle ϕ is measured counterclockwise from the x^1-coordinate axis to the y^1-coordinate axis as shown in Fig. 3. Hence, the position vector $\mathbf{R}$ may be expressed by

$$\mathbf{R}(\phi, x^3) = \bar{\mathbf{R}}(x^3) + x^\alpha \mathbf{A}_\alpha(x^3) \tag{1}$$

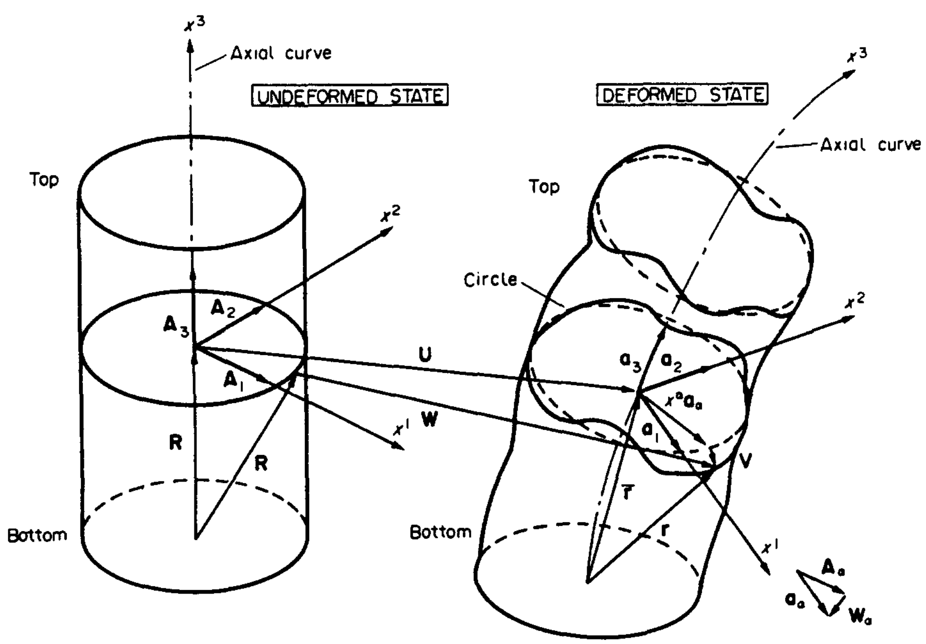

Fig. 2. Deformations of cylinders.

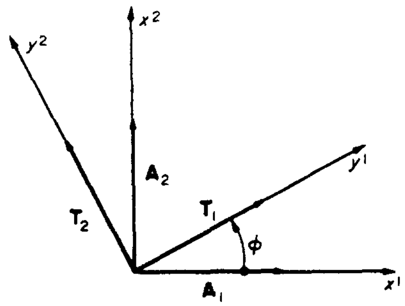

Fig. 3. Angle of ϕ.

in which the vector $\bar{\mathbf{R}}(x^3)$ is the position vector of the axial point, and the base vectors $\mathbf{A}_\alpha(x^3)$ are unit vectors taken along the transverse x^α-axes, respectively.

Now, it is assumed that the deformations of a point on the middle surface in the cylindrical shell can be expressed by the sum of the fundamental plane deformation, that can be represented sufficiently by the variations of three base vectors, $\mathbf{A}_i$, prepared on the axial x^3-coordinate, and the higher-ordered deformation occurring by warping and the distortion of the transverse cross-section on the middle surface. Also, it is assumed that the latter higher-ordered deformation adds to the fundamental deformation. The former deformation consists of the displacement vector $\mathbf{U}(x^3)$ of the axial point and the rotation deformation expressed by the rotations of the base vectors $\mathbf{A}_\alpha(x^3)$ prescribed on the axial line. The Bernoulli–Euler beam and the Timoshenko beam are contained in this deformation state. On the other hand, the latter deformation is indicated by the vector $\mathbf{V}(\phi, x^3)$.

Hence, the position vector $\mathbf{r}(\phi, x^3)$ of a point on the deformed middle surface may be written as

$$\mathbf{r}(\phi, x^3) = \bar{\mathbf{R}}(x^3) + \mathbf{U}(x^3) + x^\alpha\, \mathbf{a}_\alpha(x^3) + \mathbf{V}(\phi, x^3). \tag{2}$$

Here, the transverse base vectors $\mathbf{a}_\alpha(x^3)$ on the axial curve in the deformed state are not prescribed generally in unit vectors, and they are also not necessarily perpendicular to the base vector $\mathbf{a}_3$ on the deformed axial curve. Such constraints to the base vectors $\mathbf{a}_\alpha$ will be discussed in Section 5. The base vectors $\mathbf{a}_\alpha$ may be interpreted as directors in a broad sense[5].

Now we can express, without losing generality, the base vectors $\mathbf{a}_\alpha$ as

$$\mathbf{a}_\alpha(x^3) = \mathbf{A}_\alpha(x^3) + \mathbf{W}_\alpha(x^3) \tag{3}$$

in which the vectors $\mathbf{W}_\alpha(x^3)$ are, as shown in Fig. 2, the general rotation vectors both of which can express the rotations of the base vectors $\mathbf{A}_\alpha$ and allow the base vectors $\mathbf{a}_\alpha$ to change in length, from Refs [8, 9]. Hence, the constraints which the vectors $\mathbf{a}_\alpha$ take in the deformations can be uniquely expressed by the vectors $\mathbf{W}_\alpha(x^3)$.

Now, the position vector $\mathbf{r}$ given by eqn (2) can be written as

$$\mathbf{r}(\phi, x^3) = \mathbf{R}(\phi, x^3) + \mathbf{W}(\phi, x^3) \tag{4}$$

in which the vector $\mathbf{W}(\phi, x^3)$ is the total displacement vector of a point on the middle surface

$$\mathbf{W}(\phi, x^3) = \mathbf{U}(x^3) + x^\alpha \mathbf{W}_\alpha(x^3) + \mathbf{V}(\phi, x^3). \tag{5}$$

Then, although the base vector $\mathbf{A}_3$ on the axial line in the undeformed state is defined by

$$\mathbf{A}_3 = \bar{\mathbf{R}}_{,3} \tag{6}$$

we can assume it as a unit vector without losing generality.

The base vectors $\mathbf{G}_i$ and $\mathbf{g}_i$ of a point on the undeformed and deformed middle surfaces, respectively, can be expressed as

$$\begin{aligned} \mathbf{G}_i &= \mathbf{R}_{,i} = \mathbf{A}_i \\ \mathbf{g}_3 &= \mathbf{r}_{,3} = \mathbf{A}_3 + \mathbf{U}_{,3} + x^\alpha \mathbf{W}_{\alpha,3} + \mathbf{V}_{,3} \\ \mathbf{g}_\alpha &= \mathbf{r}_{,\alpha} = \mathbf{A}_\alpha + \mathbf{W}_\alpha + \mathbf{V}_{,\alpha}. \end{aligned} \tag{7}$$

Hence, the metric tensors G_{ij} and g_{ij} of a point on the middle surface can be obtained in linearized forms

$$\begin{aligned} G_{ij} &= \delta_{ij} \\ g_{33} &= A_{33} + 2\mathbf{A}_3(\mathbf{U}_{,3} + x^\alpha \mathbf{W}_{\alpha,3} + \mathbf{V}_{,3}) \\ g_{3\alpha} &= \mathbf{A}_3(\mathbf{W}_\alpha + \mathbf{V}_{,\alpha}) + \mathbf{A}_\alpha(\mathbf{U}_{,3} + x^\beta \mathbf{W}_{\beta,3} + \mathbf{V}_{,3}). \end{aligned} \tag{8}$$

Now, the components of displacement vectors, $\mathbf{U}$, $\mathbf{W}_\alpha$, and, $\mathbf{V}$, are defined by

$$\mathbf{U}(x^3) = U^i \mathbf{A}_i = U_i \mathbf{A}^i \tag{9}$$

$$\mathbf{W}_\alpha(x^3) = W_\alpha{}^i \mathbf{A}_i = W_{\alpha i} \mathbf{A}^i \tag{10}$$

$$\mathbf{V}(\phi, x^3) = V^i \mathbf{A}_i = V_i \mathbf{A}^i = \tilde{V}^a \mathbf{T}_a = \tilde{V}_a \mathbf{T}^a. \tag{11}$$

Also, the differentiations of the displacement vectors $\mathbf{U}$, $\mathbf{W}_\alpha$, and $\mathbf{V}$ with respect to the variable x^3 are written as

$$\mathbf{U}_{,3} = U^i_{,3} \mathbf{A}_i = U_{i,3} \mathbf{A}^i \tag{12}$$

$$\mathbf{W}_{\alpha,3} = W^i_{\alpha,3} \mathbf{A}_i = W_{\alpha i,3} \mathbf{A}^i \tag{13}$$

$$\mathbf{V}_{,3} = V^i_{,3} \mathbf{A}_i = V_{i,3} \mathbf{A}^i = \tilde{V}^a_{,3} \mathbf{T}_a = \tilde{V}_{a,3} \mathbf{T}^a. \tag{14}$$

Although the vectors $\mathbf{T}_a$ are the base vectors in the y-coordinate system prepared on the middle line of the transverse cross-section prescribed by the value of x^3 in the undeformed state, we can assume that they are unit vectors without losing generality. Hence, the relations between the base vectors $\mathbf{G}_i$ and $\mathbf{T}_a$ of a point on the middle surface can be expressed by

$$\mathbf{T}_a = C^k_a \mathbf{A}_k, \qquad \mathbf{A}_k = (C^{-1})^a_k \mathbf{T}_a \tag{15}$$

in which C^k_a and $(C^{-1})^a_k$ are rotation tensors defined by

$$(C^{-1})^a_k = C^a_k = \delta^a_\alpha \delta^\alpha_k \cos\phi + \varepsilon_{ak} \sin\phi + \delta^a_3 \delta^3_k. \tag{16}$$

Here, ε_{ak} is a permutation symbol.

Substituting eqns (15) into eqn (11), relations between the displacement components V_i and $\tilde{V}_a$ can be given by

$$V_i = \tilde{V}_a (C^{-1})^a_i. \tag{17}$$

The displacement components $\tilde{V}_a(y^1, y^2, x^3)$ with respect to the y-coordinate system are convenient for expressing both warping and the distortion of the transverse cross-section, since the physical meaning is more explicit than the displacement components V_i with respect to the x-coordinate system. Wherein the displacement component $\tilde{V}_1$ is the radial displacement of the distortion, normal to the middle surface, and the component $\tilde{V}_2$ is the circumferential displacement of the distortion. Also, the component $\tilde{V}_3$ is the warping, parallel to the axis of the shell. Now, from the symmetry of the shell with respect to the diameter plane $\phi = 0$, it may be expected from Ref. [6] that the following choice of sines and cosines for $\tilde{V}_a$ fits together:

$$\begin{aligned}\tilde{V}_1(y^\xi, x^3) &= \sum_{m=0}^{*} \overset{(m)}{\tilde{V}_1}(x^3) \cos m\phi(y^\xi)\\ \tilde{V}_2(y^\xi, x^3) &= \sum_{m=0}^{*} \overset{(m)}{\tilde{V}_2}(x^3) \sin m\phi(y^\xi)\\ \tilde{V}_3(y^\xi, x^3) &= \sum_{m=0}^{*} \overset{(m)}{\tilde{V}_3}(x^3) \cos m\phi(y^\xi)\end{aligned} \tag{18}$$

in which m denotes the circumferential wave numbers of the displacements. However, since the wave numbers, m, express the higher-ordered deformation, the $m = 1$ case is cut out. Hence, $\sum_{m=0}^{*}$ indicates the sum from $m = 0$ to ∞ except for $m = 1$. For brevity, employing the notation $\overset{(m)}{\varphi}_a$ defined by

$$\overset{(m)}{\varphi}_a = (\delta_a^1 + \delta_a^3) \cos m\phi + \delta_a^2 \sin m\phi \tag{19}$$

in eqns (18), we can obtain the following expression:

$$\tilde{V}_a(y^\xi, x^3) = \sum_{m=0}^{*} \overset{(m)}{\tilde{V}}_{[a]}(x^3)\, \overset{(m)}{\varphi}_{[a]}(y^\xi) \qquad (a\text{: no sum}). \tag{20}$$

The following notations are also defined for brevity:

$$\overset{(m)}{\varphi}{}_i^a = \overset{(m)}{\varphi}_{[a]}\, (C^{-1})_i^{[a]} \qquad (a\text{: no sum}). \tag{21}$$

Hence, substituting eqns (20) and (21) in eqn (17), the displacement components V_i can be expressed by

$$V_i = \sum_{m=0}^{*} \overset{(m)}{\tilde{V}}_a(x^3)\, \overset{(m)}{\varphi}{}_i^a. \tag{22}$$

Also, differentiations of eqn (22) with respect to the variables x^i can be written as

$$\begin{aligned}V_{i,3} &= \sum_{m=0}^{*} \overset{(m)}{\tilde{V}}_{a,3}\, \overset{(m)}{\varphi}{}_i^a\\ V_{3,\alpha} &= \sum_{m=0}^{*} \overset{(m)}{\tilde{V}}_3\, \overset{(m)}{\varphi}{}_{3,\alpha}^3\end{aligned} \tag{23}$$

where $\overset{(m)}{\varphi}{}_{3,\alpha}^3$ takes the following values:

$$\overset{(m)}{\varphi}{}_{3,\alpha}^3 = \frac{\delta_\alpha^1\, m \sin m\phi}{r \sin \phi}. \tag{24}$$

Here r is a radius of the deformed cylindrical shell.

Lastly, let us consider the rate of twist, ω. The rate of twist, ω, is given rigorously by

$$\omega = \mathbf{t}_1 \cdot \mathbf{t}_{2,3} - \mathbf{T}_1 \cdot \mathbf{T}_{2,3}. \tag{25}$$

Since the base vectors $\mathbf{t}_\xi$ ($\xi = 1, 2$) in the deformed state with respect to the y-coordinate system may be expressed as

$$\mathbf{t}_\xi = \mathbf{r}_{,\xi} = \mathbf{T}_\xi + \mathbf{V}_{,\xi} \tag{26}$$

the rate of twist, ω, can be calculated from eqn (25). But, since the expression given by this method is complicated, we use the following mean value of the rate of twist approximately, which neglects an effect of the distortion of the transverse cross-section:

$$\omega = \mathbf{a}_1 \cdot \mathbf{a}_{2,3} - \mathbf{A}_1 \cdot \mathbf{A}_{2,3}. \tag{27}$$

Substituting eqn (3) into the above equation and expressing by means of eqns (9)–(14), the rate of twist, ω, can be obtained in the following linearized form:

$$\omega = W_{21,3}. \tag{28}$$

4. STRAIN TENSORS

The Cauchy–Green strain tensors E_{3i} on the middle surface are defined by

$$E_{3i} = \tfrac{1}{2}(g_{3i} - G_{3i}). \tag{29}$$

Substituting eqn (8) into eqn (29), the linearized strain tensors E_{3i} can be written as

$$\begin{aligned} E_{33} &= U_{3,3} + x^\alpha W_{\alpha 3,3} + V_{3,3} \\ E_{3\alpha} &= \tfrac{1}{2}[U_{\alpha,3} + W_{\alpha 3} + x^\beta W_{\beta\alpha,3} + V_{\alpha,3} + V_{3,\alpha}]. \end{aligned} \tag{30}$$

Substituting eqn (23) into eqns (30), the following strain distributions, prescribed on the middle surface, are obtained:

$$\begin{aligned} E_{33} &= e_3 + x^\alpha e_{3\alpha} + \sum_{m=0}^{*} \overset{(m)}{e_3} \overset{(m)}{\varphi^3_3} \\ E_{3\alpha} &= e_\alpha + x^\beta e_{\alpha\beta} + \sum_{m=0}^{*} [\overset{(m)}{e_\xi} \overset{(m)}{\varphi^\xi_\alpha} + \overset{(m)}{e} \overset{(m)}{\varphi^3_{3,\alpha}}]. \end{aligned} \tag{31}$$

Here, the strain measures e_i, $e_{i\alpha}$, $\overset{(m)}{e_\alpha}$, and $\overset{(m)}{e}$ are defined as follows:

$$\begin{aligned} e_3 &= U_{3,3} & e_\alpha &= \tfrac{1}{2}[U_{\alpha,3} + W_{\alpha 3}] \\ e_{3\alpha} &= W_{\alpha 3,3} & e_{\alpha\beta} &= \tfrac{1}{2} W_{\beta\alpha,3} \\ \overset{(m)}{e_3} & \quad \overset{(m)}{\tilde{V}_{3,3}} & \overset{(m)}{e_\xi} &= \tfrac{1}{2} \overset{(m)}{\tilde{V}_{\xi,3}} \\ & & \overset{(m)}{e} &= \tfrac{1}{2} \overset{(m)}{\tilde{V}_3}. \end{aligned} \tag{32}$$

5. GOVERNING EQUATIONS OF MOTION

Let us present the governing equations of motion to circular cylindrical shells by means of the principle of virtual work. Since the strain tensors given by eqn (29) are the mean strain tensors prescribed on the middle surface, the effect of the thickness-wise variation of strains is neglected. This effect can be considered by adding the virtual work done by the St. Venant torsional moment, T, to the internal virtual work.

Hence, the internal virtual work δW_i may be expressed by

$$\delta W_i = \int_0^l \int_S [S^{33}\, \delta E_{33} + 2S^{3\alpha}\, \delta E_{3\alpha}]\, \mathrm{d}S\, \mathrm{d}x^3 + \int_0^l T\, \delta\omega\, \mathrm{d}x^3 \tag{33}$$

in which S^{3i} are the Kirchhoff stress tensors. The substitution of eqn (31) into the above equation gives

$$\delta W_i = \int_0^l \left[N^3 \,\delta e_3 + M^{\alpha 3}\,\delta e_{3\alpha} + 2N^{\alpha}\,\delta e_{\alpha} + 2M^{\beta\alpha}\,\delta e_{\alpha\beta} + T\,\delta\omega + \sum_{m=0}^{*} \left\{ \overset{(m)}{N^3}\,\delta \overset{(m)}{e_3} + 2\overset{(m)}{N^{\xi}}\,\delta \overset{(m)}{e_{\xi}} + 2\overset{(m)}{N}\,\delta \overset{(m)}{e} \right\} \right] \mathrm{d}x^3 \tag{34}$$

in which the stress resultants and stress couples are defined as follows:

$$\begin{Bmatrix} N^i \\ M^{\alpha i} \\ \overset{(m)}{N^a} \end{Bmatrix} = \int S^{3i} \begin{Bmatrix} 1 \\ x^{\alpha} \\ \overset{(m)}{\varphi_i^a} \end{Bmatrix} \mathrm{d}S$$

$$\overset{(m)}{N} = \int S^{3\alpha}\, \overset{(m)}{\varphi^3_{3,\alpha}}\, \mathrm{d}S. \tag{35}$$

Also, using the rate of twist given by eqn (28), the St. Venant torsional moment, T, can be expressed as

$$T = GJ\omega = GJW_{21,3} \tag{36}$$

in which the St. Venant torsional constant, J, for circular cylindrical shells takes the form

$$J = \tfrac{2}{3}\pi R t^3 + 2\pi R^3 t \simeq 2\pi R^3 t \tag{37}$$

in which R is the radius of the undeformed shell. Now, substituting eqns (28) and (32) into eqns (35), the internal virtual work, δW_i, can be written as

$$\delta W_i = \int_0^l \left[-N^i_{,3}\,\delta U_i - (M^{\alpha 3}{}_{,3} - N^{\alpha})\delta W_{\alpha 3} - M^{\alpha\beta}{}_{,3}\,\delta W_{\alpha\beta} - T_{,3}\,\delta W_{21} - \sum_{m=0}^{*} \left(\overset{(m)}{N^a_{,3}}\,\delta \overset{(m)}{\tilde{V}_a} - \overset{(m)}{N}\,\delta \overset{(m)}{\tilde{V}_3} \right) \right] \mathrm{d}x^3 + \left[N^i\,\delta U_i + M^{\alpha 3}\,\delta W_{\alpha 3} + M^{\alpha\beta}\,\delta W_{\alpha\beta} + T\,\delta W_{21} + \sum_{m=0}^{*} \overset{(m)}{N^a}\,\delta \overset{(m)}{\tilde{V}_a} \right]_{x^3=0}^{x^3=l}. \tag{38}$$

Meanwhile, the external virtual work, δW_e, is given by

$$\delta W_e = \int_V \rho(\mathbf{p}-\mathbf{c})\delta\mathbf{W}\,\mathrm{d}V + \int_{S_\sigma} \overset{*}{\mathbf{T}}\,\delta\mathbf{W}\,\mathrm{d}S + \int_{S_{\sigma_c}} \overset{*}{\mathbf{T}}\,\delta\mathbf{W}\,\mathrm{d}S \tag{39}$$

in which $\mathbf{p}(x^{\alpha}, x^3)$ and $\mathbf{c}(x^{\alpha}, x^3)$ are the external force and acceleration per unit area in the undeformed shell with density, ρ, respectively. Also, V denotes the total volume. The notation S_{σ} indicates the lateral surface of the shell on which the stress vector $\overset{*}{\mathbf{T}}$ is prescribed, and S_{σ_c} indicates the boundary surface at the end point of the shell on which the stress vector $\overset{*}{\mathbf{T}}$ is prescribed.

The components of the vectors **p**, **c**, and $\overset{*}{\mathbf{T}}$ are expressed as follows:

$$\mathbf{p} = P^i\mathbf{A}_i = P_i\mathbf{A}^i = \tilde{P}^a\mathbf{T}_a = \tilde{P}_a\mathbf{T}^a$$
$$\mathbf{c} = C^i\mathbf{A}_i = C_i\mathbf{A}^i = \tilde{C}^a\mathbf{T}_a = \tilde{C}_a\mathbf{T}^a \tag{40}$$
$$\overset{*}{\mathbf{T}} = \overset{*}{T}{}^i\mathbf{A}_i = \overset{*}{T}_i\mathbf{A}^i = \overset{*}{\tilde{T}}{}^a\mathbf{T}_a = \overset{*}{\tilde{T}}_a\mathbf{T}^a.$$

Now, the external components working with the displacement components U_i or $W_{\alpha i}$ are expressed with respect to the x-coordinate system, and the external components working with the displacements $\tilde{V}_a$ are indicated with respect to the y-coordinate system. Hence, the external virtual work can be written as

$$\delta W_e = \int_0^l \left[(p^i - c^i)\,\delta U_i + (m^{\alpha i} - d^{\alpha i})\,\delta W_{\alpha i} + \sum_{m=0}^{*}\left(\overset{(m)}{\tilde{p}^a} - \overset{(m)}{\tilde{c}^a}\right)\delta\overset{(m)}{\tilde{V}_a}\right]\mathrm{d}x^3 + \left[\overset{*}{T}{}^i\,\delta U_i + \overset{*}{m}{}^{\alpha i}\,\delta W_{\alpha i} + \sum_{m=0}^{*}\overset{(m)}{\tilde{T}^{a*}}\,\delta\overset{(m)}{\tilde{V}_a}\right]\Bigg|_{S_{\sigma_c}}. \tag{41}$$

Here, the external forces and accelerations and others are defined as

$$\begin{Bmatrix} p^i \\ m^{\alpha i}\end{Bmatrix} = \int_S \rho\,\bar{P}^i\begin{Bmatrix}1\\ x^\alpha\end{Bmatrix}\mathrm{d}S + \int_C \overset{*}{T}{}^i\begin{Bmatrix}1\\ x^\alpha\end{Bmatrix}\mathrm{d}c$$

$$\overset{(m)}{\tilde{p}^a} = \int_S \rho\,\tilde{P}^{[a]}\,\overset{(m)}{\varphi}_{[a]}\,\mathrm{d}S + \int_C \overset{*}{\tilde{T}}{}^{[a]}\,\overset{(m)}{\varphi}_{[a]}\,\mathrm{d}c \qquad (a\text{: no sum}) \tag{42}$$

$$\begin{Bmatrix} c^i \\ d^{\alpha i}\end{Bmatrix} = \int_S \rho\,\bar{C}^i\begin{Bmatrix}1\\ x^\alpha\end{Bmatrix}\mathrm{d}S$$

$$\overset{(m)}{\tilde{c}^a} = \int_S \rho\,\tilde{C}^{[a]}\,\overset{(m)}{\varphi}_{[a]}\,\mathrm{d}S \qquad (a\text{: no sum})$$

in which dc indicates the circumferential infinitesimal length of the shell. The components p^i are the components of the external load expressed in the macroscopic meaning. On the other hand, the components $\overset{(m)}{\tilde{p}^a}$ are the components of the external load, which work with either warping or the distortion of the transverse cross-section, expressed in the microscopic meaning. For example, for liquid storage tanks the components $\overset{(m)}{\tilde{p}^a}$ are given by integrating the product of the dynamic liquid pressures, $\overset{*}{\tilde{T}}{}^a$, acting inside the tank given by Housner[4] and the modes of the distortion of the transverse cross-section, $\overset{(m)}{\varphi}_a$.

Substituting eqns (38) and (41) into the principle of virtual work given by

$$\delta W_i - \delta W_e = 0 \tag{43}$$

then the equations of motion and the boundary conditions can be obtained as follows:

equations of motion

$$\delta U_i:\quad N^i_{,3} + p^i = c^i \tag{44}$$

$$\delta W_{\alpha i}:\quad M^{\alpha i}{}_{,3} - \delta^3_i\,N^\alpha + T_{,3}\,\delta^2_\alpha\,\delta^1_i + m^{\alpha i} = d^{\alpha i} \tag{45}$$

$$\delta\overset{(m)}{\tilde{V}_a}:\quad \overset{(m)}{N^a}{}_{,3} - \delta^3_a\overset{(m)}{N} + \overset{(m)}{\tilde{p}^a} = \overset{(m)}{\tilde{c}^a}; \tag{46}$$

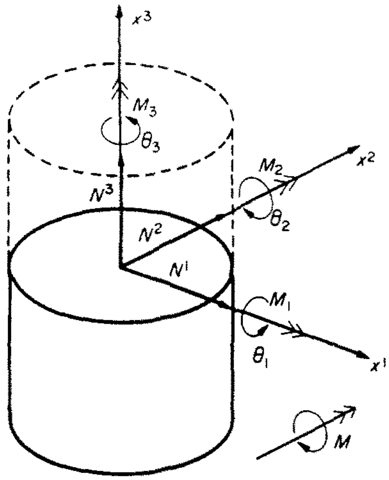

Fig. 4. Positive direction of stress resultants and stress couples.

boundary conditions

$$U_i = 0 \quad \text{or} \quad \pm N^i = \overset{*}{T}{}^i \tag{47}$$

$$W_{\alpha i} = 0 \quad \text{or} \quad \pm M^{\alpha i} + T\,\delta^{\alpha}_{2}\,\delta^{i}_{1} = \overset{*}{m}{}^{\alpha i} \tag{48}$$

$$\overset{(m)}{\tilde{V}}_a = 0 \quad \text{or} \quad \pm \overset{(m)}{N}{}^a = \overset{(m)}{\tilde{T}}{}^{a*}. \tag{49}$$

Here plus and minus signs are taken to the boundary surfaces prescribed on $x^3 = l$ and 0, respectively. If each positive direction of the stress resultants and stress couples at $x^3 = 0$ is defined in contrast with the direction given by Fig. 4, then these signs always have the plus sign.

Although the stress couples, $M^{\alpha j}$, external moments, $m^{\alpha j}$, and acceleration moments, $b^{\alpha j}$, used in the present theory are expressed in the dual form, these expressions are related to the usual expressions for moments using rod theory as follows:

$$\begin{aligned} M_i &= M^{\alpha j}\varepsilon_{\alpha ji} \\ m_i &= m^{\alpha j}\varepsilon_{\alpha ji} \\ b_i &= b^{\alpha j}\varepsilon_{\alpha ji} \end{aligned} \tag{50}$$

in which $\varepsilon_{\alpha ij}$ is a three-dimensional permutation symbol. The moments are defined as clockwise moments in the positive directions of the coordinate axes, as shown in Fig. 4.

6. CONSTITUTIVE EQUATIONS

Supposing that the strains are infinitesimal, the stress–strain relations can use the following engineering forms:

$$\begin{aligned} S^{33} &= E\,E_{33} \\ S^{3\alpha} &= 2G\,E_{3\alpha} \end{aligned} \tag{51}$$

where E is Young's modulus of elasticity, and G the shearing modulus.

Substituting eqns (51) and (31) into eqns (35), the constitutive equations for the circular cylindrical shell can be written as

$$
\begin{aligned}
N^3 &= E\left[A\, e_3 + \sum_{k=0}^{*} \underline{\overset{(k)}{I^3_3}\ \overset{(k)}{e_3}}\right]\\
N^\alpha &= 2G\left[A\, e_\alpha + \sum_{k=0}^{*} \left\{\underline{\overset{(k)}{I^\xi_\alpha}\ \overset{(k)}{e_\xi}} + \overset{(k)}{I^3_\alpha}\ \overset{(k)}{e}\right\}\right]\\
M^{\alpha 3} &= E I^{\alpha\beta} e_{3\beta}\\
M^{\beta\alpha} &= 2G\left[I^{\beta\eta} e_{\alpha\eta} + \sum_{k=0}^{*} \left\{\overset{(k)}{I^{\xi\beta}_\alpha}\ \overset{(k)}{e_\xi} + \overset{(k)}{I^{3\beta}_\alpha}\ \overset{(k)}{e}\right\}\right] \qquad (52)\\
\overset{(m)}{N^3} &= E\left[\underline{\overset{(m)}{I^3_3} e_3} + \sum_{k=0}^{*} \overset{(m,k)}{I^{33}}\ \overset{(k)}{e_3}\right]\\
\overset{(m)}{N^\xi} &= 2G\left[\delta^{\alpha\lambda}\left\{\underline{\overset{(m)}{I^\xi_\alpha} e_\lambda} + \overset{(m)}{I^{\xi\beta}_\alpha} e_{\lambda\beta}\right\} + \sum_{k=0}^{*} \left\{\overset{(m,k)}{I^{\xi\eta}}\ \overset{(k)}{e_\eta} + \overset{(k,m)}{I^{3\xi}}\ \overset{(k)}{e}\right\}\right]\\
\overset{(m)}{N} &= 2G\left[\delta^{\alpha\lambda}\left\{\overset{(m)}{I^3_\alpha} e_\lambda + \overset{(m)}{I^{3\beta}_\alpha} e_{\lambda\beta}\right\} + \sum_{k=0}^{*} \left\{\overset{(m,k)}{I^{3\xi}}\ \overset{(k)}{e_\xi} + \underline{\overset{(m,k)}{I}\ \overset{(k)}{e}}\right\}\right].
\end{aligned}
$$

Here, the constants of the cross-section are defined as

$$
\begin{aligned}
&\begin{Bmatrix} A \\ I^\alpha \\ I^{\alpha\beta} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ x^\alpha \\ x^\alpha x^\beta \end{Bmatrix} \mathrm{d}S\\
&\begin{Bmatrix} \overset{(m)}{I^3_3} \\ \overset{(m)}{I^{3\alpha}_3} \\ \overset{(m,k)}{I^{33}} \end{Bmatrix} = \int \overset{(m)}{\varphi^3_3} \begin{Bmatrix} 1 \\ x^\alpha \\ \overset{(k)}{\varphi^3_3} \end{Bmatrix} \mathrm{d}S, \quad
\begin{Bmatrix} \overset{(m)}{I^a_\alpha} \\ \overset{(m)}{I^{a\beta}_\alpha} \\ \overset{(m,k)}{I^{a\eta}} \end{Bmatrix} = \int \left(\delta^a_3\ \overset{(m)}{\varphi^3_{3,\alpha}} + \delta^a_\xi\ \overset{(m)}{\varphi^\xi_\alpha}\right) \begin{Bmatrix} 1 \\ x^\beta \\ \overset{(k)}{\varphi^\eta_\beta}\delta^{\alpha\beta} \end{Bmatrix} \mathrm{d}S \qquad (53)\\
&\overset{(m,k)}{I} = \int \overset{(m)}{\varphi^3_{3,\alpha}}\ \overset{(k)}{\varphi^3_{3,\beta}}\, \delta^{\alpha\beta}\, \mathrm{d}S.
\end{aligned}
$$

The constants of the cross-sections are influenced by the distortion of the transverse cross-section but are independent of warping. This influence can be considered as follows: the infinitesimal area, dS, in eqns (53) may be expressed as

$$
\mathrm{d}S = r\, \mathrm{d}\phi\, \mathrm{d}r. \qquad (54)
$$

Here r is a radius of the deformed shell, as shown in Fig. 5, and is a function of the distortion of the transverse cross-section.

It is sufficient to consider only transverse loads acting in one direction for usual external loads acting on cantilevered circular cylindrical shells used as tanks and silos. Accordingly, let us assume that the external loads are transverse loads acting in the x^1-direction only. Then, in the distortion of the transverse cross-section the radial displacement $\overset{(n)}{\tilde{V}_1}$ becomes important and the circumferential displacement $\overset{(n)}{\tilde{V}_2}$ is of a negligible order as compared with the displacement $\overset{(n)}{\tilde{V}_1}$. Hence, the radius r can be expressed approximately as

$$
\begin{aligned}
r(\phi, x^3) &= \sqrt{((R+\tilde{V}_1)^2 + (\tilde{V}_2)^2)} \simeq R + \tilde{V}_1\\
&= R + \sum_{n=0}^{*} \overset{(n)}{\tilde{V}_1} \cos n\phi.
\end{aligned} \qquad (55)
$$

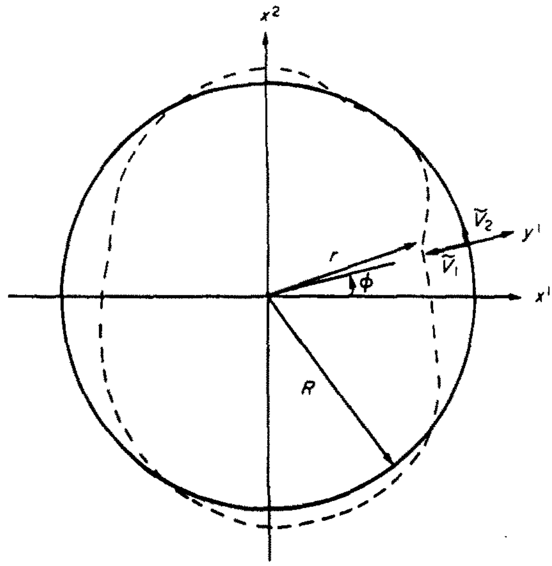

Fig. 5. Distortion of the cross-section.

Substituting eqn (55) into eqn (54), we can write

$$\mathrm{d}S = \left(R + \sum_{n=0}^{*} \overset{(n)}{\tilde V_1} \cos n\phi\right) \mathrm{d}\phi\, \mathrm{d}r \simeq t\left(R + \sum_{n=0}^{*} \overset{(n)}{\tilde V_1} \cos n\phi\right) \mathrm{d}\phi. \tag{56}$$

Hence, the constants of the cross-section calculated by eqn (56) can be expressed, respectively, by the sum of the usual constant of the cross-section without the distortion of the transverse cross-section, which is indicated with subscript 0, and the constant of the cross-section with the distortion of the transverse cross-section of the circumferential wave numbers, n, which is indicated with superscript *. For example

$$I^{\alpha\beta} = {}_0I^{\alpha\beta} + \sum_{n=0}^{*} \overset{(n)}{\tilde V_1} \overset{*}{I}{}^{\alpha\beta}(n). \tag{57}$$

Provided that the cross-sectional area, A, is independent of distortion.

However, since the relation $\tilde V_1 \ll R$ is always valid for usual circular cylindrical shells, the influence of the distortion of the transverse cross-section on the constants of the cross-section is negligible. Hence, this influence is considered only for the geometrical moment of inertia and is neglected for the others. The values of the constants of the cross-section are given in the Appendix. Also, using this approximation, the underlined terms in the constitutive equations, eqns (52), vanish.

7. APPROXIMATIONS

The governing equations of rod theory including the distortion of the transverse cross-section to the circular cylindrical shells have been given by the strain–displacement relations (32), the equations of motion (44)–(46), the constitutive equations (52), and the boundary conditions (45)–(49), for displacement components U_i, $W_{\alpha i}$, and $\overset{(m)}{\tilde V_a}$. To apply the presented theory to practical problems, it is necessary to decrease the unknown quantities. This is accomplished by using the classical hypotheses in rods.

As for the classical hypotheses on rod theory, ignoring both warping and the distortion of the transverse cross-section, there are the Bernoulli–Euler hypothesis and the Timoshenko beam hypothesis.

The Bernoulli–Euler hypothesis consists of the rigid displacement of the cross-section

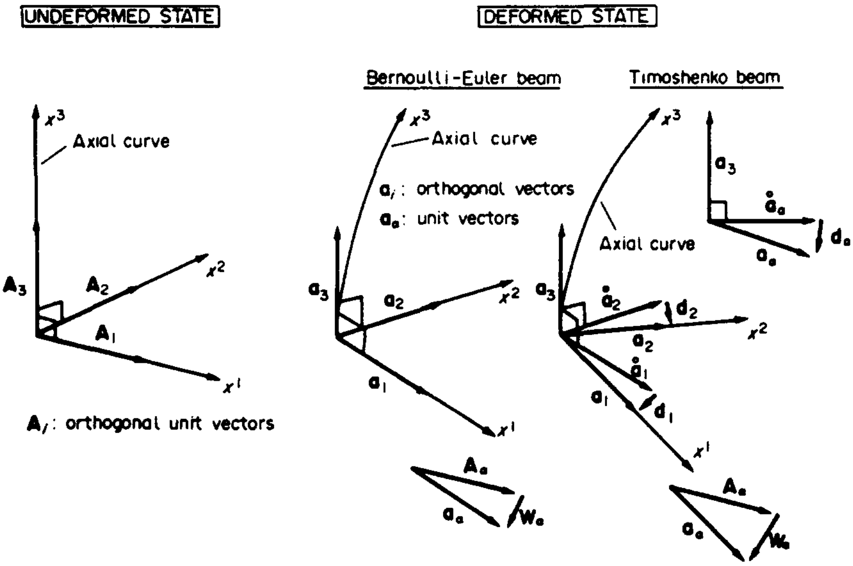

Fig. 6. Bernoulli–Euler beam and Timoshenko beam.

and the conservation of the perpendicularity of the cross-section, as shown in Fig. 6. The Bernoulli–Euler hypothesis in the present theory implies that the following relations for the base vectors $\mathbf{a}_i$ on the deformed axial curve must exist:

$$\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \delta_{\alpha\beta} \tag{58}$$

$$\mathbf{a}_3 \cdot \mathbf{a}_\alpha = 0. \tag{59}$$

Since the base vector, $\mathbf{a}_3$, on the deformed axial curve is expressed by

$$\mathbf{a}_3 = \bar{\mathbf{r}}_{,3} = \mathbf{A}_3 + \mathbf{U}_{,3} \tag{60}$$

eqns (58) and (59) can be rewritten in linearized forms as

$$W_{\alpha\beta} + W_{\beta\alpha} = 0 \tag{61}$$

$$W_{\alpha 3} + U_{\alpha,3} = 0. \tag{62}$$

Hence, the Bernoulli–Euler hypothesis in the infinitesimal deformation can be expressed by

$$W_{11} = W_{22} = 0, \qquad W_{12} = -W_{21} \tag{63}$$

$$W_{\alpha 3} = -U_{\alpha,3}. \tag{64}$$

On the other hand, the Timoshenko beam hypothesis extends the Bernoulli–Euler hypothesis, such as considering the transverse shear deformation by the mean transverse shear deformation. Therefore, the base vectors $\mathbf{a}_3$ and $\mathbf{a}_\alpha$ on the deformed axial curve do not perpendicularly intersect each other. Now, let us express the base vectors $\mathbf{a}_\alpha$ in the deformed state by

$$\mathbf{a}_\alpha(x^3) = \mathring{\mathbf{a}}_\alpha(x^3) + \mathbf{d}_\alpha(x^3) \tag{65}$$

in which vectors $\mathring{\mathbf{a}}_\alpha$ are base vectors perpendicular to the base vector $\mathbf{a}_3$, as shown in Fig. 6, and they coincide with the base vectors in the Bernoulli–Euler beam. Also, the vectors $\mathbf{d}_\alpha(x^3)$ indicate the mean transverse shear deformations. Hence, the Timoshenko beam hypothesis can be expressed by

$$\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \delta_{\alpha\beta} \tag{66}$$

$$\mathring{\mathbf{a}}_\alpha \cdot \mathbf{a}_3 = 0. \tag{67}$$

Expressing the above equations by displacements we can take the linearized forms

$$W_{11} = W_{22} = 0, \qquad W_{12} = -W_{21} \tag{68}$$

$$W_{\alpha 3} = -U_{\alpha,3} + D_{\alpha 3} \tag{69}$$

in which $D_{\alpha i}$ are the components of the vectors $\mathbf{d}_\alpha$. For usual structures, it is sufficient to consider only the displacement components $D_{\alpha 3}$ for $\mathbf{d}_\alpha$. Hence, we can assume

$$\mathbf{d}_\alpha = D_{\alpha i}\mathbf{A}^i \simeq D_{\alpha 3}\mathbf{A}^3. \tag{70}$$

The above-mentioned two hypotheses ignore both warping and the distortion of the transverse cross-section. As for a typical theory including warping, the Vlasov theory is well known. Since the Vlasov theory considers only warping in addition to the Bernoulli–Euler hypothesis prescribed by eqns (58) and (59), the displacement, $\mathbf{V}$, can be written as

$$\mathbf{V} = \tilde{V}^3\mathbf{T}_3 = V^3\mathbf{A}_3, \qquad \tilde{V}_\xi = \tilde{V}^\xi = 0. \tag{71}$$

Then, the relations between the rotational angles, θ^i, used generally in rod theory and the rotation tensors $W_{\alpha 3}$ and W_{12} $(= -W_{21})$ used in the present theory can be expressed by

$$W_{\alpha i} = \varepsilon_{\alpha ij}\,\theta^j \tag{72}$$

as rigid displacement of the cross-section is assumed. Here θ^j are the rotational angles about the x^j-coordinate axes, as shown in Fig. 4, and their positive directions are defined as the clockwise rotation with respect to the coordinate axes.

Also, in rods the following expression which eliminates the shearing forces N^α from eqns (44) and (45) is used generally:

$$M^{\alpha 3}{}_{,33} + m^{\alpha 3}{}_{,3} + p^\alpha = \underline{d^{\alpha 3}{}_{,3}} + c^\alpha. \tag{73}$$

Since the differential term of the acceleration moment is generally negligible, we can take the form

$$M^{\alpha 3}{}_{,33} + m^{\alpha 3}{}_{,3} + p^\alpha = c^\alpha. \tag{74}$$

8. EXAMPLES

In order to examine the presented theory, a static and an elastic problem of a cantilevered circular cylindrical shell, as shown in Fig. 7, is considered. The shell bottom is regarded as anchored to its rigid foundation, and the top of the shell is assumed to be open. Also, it is assumed that a load is only a uniform transverse load per unit length of the x^1-coordinate axis acting at a position at $\phi = 180°$ in the cross-section. Let us analyze this problem using the Bernoulli–Euler beam theory, the Timoshenko beam theory, and the theory including the distortion of the transverse cross-section, which these theories are obtained from the derived governing equations. For brevity, the x^3-coordinate axis in the present section is indicated by x.

8.1. *The Bernoulli–Euler beam*

Using the Bernoulli–Euler hypothesis, the unknown displacement for the present problem becomes either U_1 or θ^2 $(= -W_{12})$ alone. Hence, from eqn (74), the equilibrium

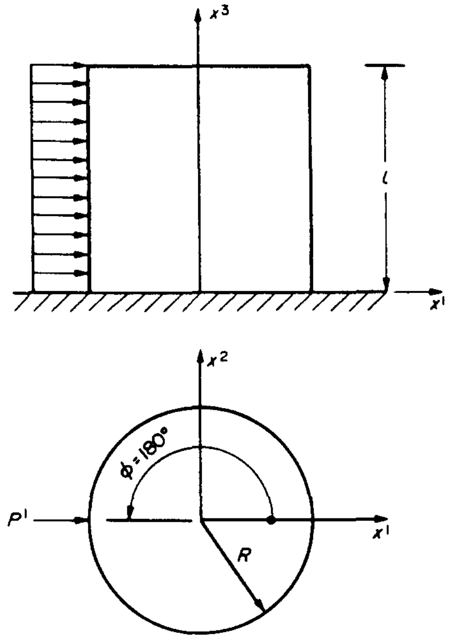

Fig. 7. Cantilevered cylinders.

equation can be written as

$$M^{13}{}_{,33}+p^1 = 0. \tag{75}$$

By substituting eqn (32) into eqn (52), the stress couple M^{13} is expressed as

$$M^{13} = EI^{11}W_{13,3} = -EI^{11}\theta^2{}_{,3}. \tag{76}$$

By ignoring the transverse shear deformation, the rotation angle θ^2 can be expressed as

$$\theta^2 = U_{1,3}. \tag{77}$$

Hence, the equilibrium equation, eqn (75), becomes

$$U_{1,3333} = \frac{p^1}{EI^{11}}. \tag{78}$$

The general solution of the above equation can be obtained as follows:

$$U_1 = C_1x^3+C_2x^2+C_3x+C_4+\frac{p^1x^4}{24EI^{11}} \quad \text{where} \quad x = x^3. \tag{79}$$

The integral constants C_1, C_2, C_3, and C_4 are determined from the following boundary conditions:

$$U_1 = 0 \quad \text{at} \quad x = 0 \tag{80}$$

$$\theta^2 = 0 \quad \text{at} \quad x = 0 \tag{81}$$

$$M^{13}{}_{,3}-N^1 = 0 \quad \text{at} \quad x = l \tag{82}$$

$$M^{13} = 0 \quad \text{at} \quad x = l. \tag{83}$$

Hence, the solution can be obtained as

$$U_1 = \frac{p^1 l^4}{24EI^{11}}[6\bar{x}^2 - 4\bar{x}^3 + \bar{x}^4] \qquad \text{where} \quad \bar{x} = x/l$$
$$\theta^2 = U_{1,3} = \frac{p^1 l^3}{6EI^{11}}[3\bar{x} - 3\bar{x}^2 + \bar{x}^3]. \tag{84}$$

And, the stress couple, M_2, and the stress resultant, N^1, yield

$$M_2 = -M^{13} = \frac{p^1 l^2}{2}(1-\bar{x})^2 \tag{85}$$

$$N^1 = M^{13}{}_{,3} = p^1 l(1-\bar{x}). \tag{86}$$

These results agree with the well-known results of the Bernoulli–Euler beam.

8.2. *The Timoshenko beam*

Considering the Timoshenko beam, the unknown displacements are U_1 and either W_{13} or D_{13}. From eqn (59), the displacement W_{13} is given by

$$W_{13} = -U_{1,3} + D_{13}. \tag{87}$$

Expressing eqns (44) and (74) by the displacements, the equilibrium equations can be written as

$$G\kappa A(W_{13,3} + U_{1,33}) + p^1 = 0 \tag{88}$$

$$EI^{11} W_{13,333} + p^1 = 0 \tag{89}$$

in which κ is a factor considering the distribution of the shearing stress, and for example, it can employ the following value given by Cowper[7]:

$$\kappa = \frac{(7+6\nu)(1+n^2)^2 + (20+12\nu)n^2}{6(1+\nu)(1+n^2)^2} \tag{90}$$

where n is a ratio of the external to internal diameters, and ν is Poisson's ratio.

Substituting eqn (87) into eqns (88) and (89), the general solutions for U_1 and D_{13} can be obtained as follows:

$$U_1 = C_1 x^3 + C_2 x^2 + C_3 x + C_4 + \frac{p^1 x^4}{24EI^{11}} \tag{91}$$

$$D_{13} = -\frac{p^1 x}{G\kappa A} + C_5 \tag{92}$$

in which the integral constants C_1, C_2, C_3, C_4, and C_5 are determined by the following boundary conditions:

$$U_1 = 0 \quad \text{at} \quad x = 0 \tag{93}$$

$$W_{13} = 0 \quad \text{at} \quad x = 0 \tag{94}$$

$$N^1 = 0 \quad \text{at} \quad x = l \tag{95}$$

$$EI^{11} W_{13,33} - G\kappa A(W_{13} + U_{1,3}) = 0 \quad \text{at} \quad x = l \tag{96}$$

$$M^{13} = 0 \quad \text{at} \quad x = l. \tag{97}$$

Hence, the above solutions can be written as

$$U_1 = \frac{p^1 l^4}{24EI^{11}}(6\bar{x}^2 - 4\bar{x}^3 + \bar{x}^4) + \frac{p^1 l^2}{2G\kappa A}\bar{x}(2-\bar{x})$$
$$D_{13} = \frac{p^1 l}{G\kappa A}(1-\bar{x}). \tag{98}$$

Also, the displacement $W_{13}\,(=-\theta_2)$, the stress couple, and the shearing force are the same as eqns (84), (85), and (86), respectively, given in the Bernoulli–Euler beam. The obtained results coincide with the well-known results of the Timoshenko beam theory.

8.3. *The theory including the distortion of the transverse cross-section*

Let us ignore the transverse shear deformation for brevity. Also, since the transverse load considered does not have a torsional moment, the warping $\overset{(m)}{\tilde{V}_3}$ is negligible. Hence, it is sufficient to consider only U_1 and $\overset{(m)}{\tilde{V}_\xi}$ ($\xi = 1, 2$) for displacements. From eqns (74) and (46), the equilibrium equations for these displacements can be written as

$$\delta U_1:\quad E\left[{}_0I^{11} + \overset{*}{\sum_{n=0}} \overset{(n)}{\tilde{V}_1}\,\mathring{I}^{11}(n)\right] W_{13,333} + p^1 = 0 \tag{99}$$

$$\delta \overset{(m)}{\tilde{V}_\xi}:\quad G\overset{*}{\sum_{k=0}} \overset{(m,k)}{I^{\xi\eta}}\,\overset{(k)}{\tilde{V}_{\eta,33}}\,\overset{(m)}{\tilde{p}^\xi} = 0. \tag{100}$$

From the Appendix, the constants of the cross-section can be given by

$${}_0I^{11} \simeq \pi t R^3,\qquad \mathring{I}^{11}(n) = \tfrac{3}{2}\pi t R^2\,\delta(n-2),\qquad \overset{(m,k)}{I^{\xi\eta}} \simeq \pi t R\,\delta^{\xi\eta}\,\delta(n-k) \tag{101}$$

in which the notation $\delta(n-k)$ is given by eqn (A2) in the Appendix. Considering eqns (101), eqn (100) can be rewritten in the form

$$\delta \overset{(m)}{\tilde{V}_\xi}:\quad G\,\overset{(m,m)}{I^{\xi\xi}}\,\overset{(m)}{\tilde{V}_{\xi,33}} + \overset{(m)}{\tilde{p}^\xi} = 0\qquad (m \text{ and } \xi\text{: no sum}). \tag{102}$$

Owing to a lowering of the geometrical moment of inertia, I^{11}, by the distortion of the transverse cross-section, eqn (99) is of the coupled form.

Now, since the load $\overset{(m)}{\tilde{p}^\xi}$ is independent of the variable x^3, the general solutions for eqn (102) are

$$\overset{(m)}{\tilde{V}_\xi} = C_1 x + C_2 - \frac{\overset{(m)}{\tilde{p}^\xi}}{G\,\overset{(m,m)}{I^{\xi\xi}}}\frac{x^2}{2}\qquad (m \text{ and } \xi\text{: no sum}). \tag{103}$$

The integral constants C_1 and C_2 are determined from the boundary conditions. The boundary conditions for the present problem can be written as follows:

$$U_1 = 0 \quad \text{at}\quad x = 0 \tag{104}$$

$$W_{13} = -U_{1,3} = 0 \quad \text{at}\quad x = 0 \tag{105}$$

$$\overset{(m)}{\tilde{V}_\xi} = 0 \quad \text{at}\quad x = 0 \tag{106}$$

$$M^{13}{}_{,3} - N^1 = 0 \quad \text{at}\quad x = l \tag{107$_1$}$$

$$M^{13} = 0 \quad \text{at}\quad x = l \tag{108$_1$}$$

$$\overset{(m)}{N^\xi} = 0 \quad \text{at}\quad x = l. \tag{109$_1$}$$

Equations $(107)_1$–$(109)_1$ can be rewritten, respectively, in displacements as follows:

$$U_{1,333} = 0 \quad \text{at} \quad x = l \tag{107$_2$}$$

$$U_{1,33} = 0 \quad \text{at} \quad x = l \tag{108$_2$}$$

$$\overset{(m)}{\tilde{V}}_{\xi,3} = 0 \quad \text{at} \quad x = l. \tag{109$_2$}$$

Determining the integral constants from eqns (106) and $(109)_2$, the displacements $\overset{(m)}{\tilde{V}}_\xi$ can be given as follows:

$$\overset{(m)}{\tilde{V}}_\xi = \frac{\overset{(m)}{\tilde{p}^\xi}\, l^2}{G \overset{(m,m)}{I^{\xi\xi}}} \frac{\bar{x}}{2}(2-\bar{x}) \qquad (m \text{ and } \xi : \text{no sum}). \tag{110}$$

Then, let us try to obtain the displacement U_1. Equation (99) can be rewritten as

$$E\left[{}_0I^{11} + \sum_{n=0}^{*} \overset{(n)}{\tilde{V}}_1 \overset{*}{I}{}^{11}(n)\right] U_{1,3333} - p^1 = 0 \tag{111}$$

with U_1. The term in square brackets is given in the form

$${}_0I^{11} + \sum_{n=0}^{*} \overset{(n)}{\tilde{V}}_1 \overset{*}{I}{}^{11}(n) = {}_0I^{11}\left[1 + \overset{(2)}{\tilde{V}}_1 \frac{3}{2R}\right] \tag{112}$$

which is a function of x^3. Although it is possible to obtain analytically the solution of eqn (111), the following assumption is employed to obtain easily the solutions: since $\overset{(2)}{\tilde{V}}_1/R \ll 1$ for usual structures, eqn (111) can be rewritten approximately as

$$U_{1,3333} = \frac{p^1}{E\,{}_0I^{11}\left[1 + \overset{(2)}{\tilde{V}}_1 \frac{3}{2R}\right]} \simeq \frac{p^1}{E\,{}_0I^{11}}\left[1 - \overset{(2)}{\tilde{V}}_1 \frac{3}{2R}\right]. \tag{113}$$

The substitution of eqn (100) is

$$U_{1,3333} = a + bx + cx^2 \tag{114}$$

in which a, b, and c take the following values:

$$a = \frac{p^1}{E\,{}_0I^{11}}, \quad b = -\frac{p^1}{E\,{}_0I^{11}} \frac{3\overset{(2)}{\tilde{p}^1}\, l}{2G\pi t R^2}, \quad c = \frac{p^1}{E\,{}_0I^{11}} \frac{3\overset{(2)}{\tilde{p}^1}}{4G\pi t R^2}. \tag{115}$$

The general solution of eqn (114) is

$$U_1 = \frac{ax^4}{24} + \frac{bx^5}{120} + \frac{cx^6}{360} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4. \tag{116}$$

Determining the integral constants C_1 to C_4 from boundary conditions (104), (105), $(107)_2$, and $(108)_2$, the displacement U_1 is given by

$$U_1 = \frac{p^1 l^4}{24E\,{}_0I^{11}}[6\bar{x}^2 - 4\bar{x}^3 + \bar{x}^4] + \frac{b\, l^5}{120}[20\bar{x}^2 - 10\bar{x}^3 + \bar{x}^5] + \frac{c\, l^6}{360}[45\bar{x}^2 - 20\bar{x}^3 + \bar{x}^6]. \tag{117}$$

The first term in eqn (117) indicates the transverse deflection of the Bernoulli–Euler beam. The underlined terms indicate the transverse deflection of the axial curve x^3 caused by a lowering of the bending stiffness by the distortion of the transverse cross-section.

Now, taking the size of the circular cylindrical shell and the load p^1 as follows:

$$\text{radius } R = 500 \text{ cm}, \quad \text{height } l = 1000 \text{ cm}, \quad \text{thickness } t = 0.3 \text{ cm},$$
$$E = 2.1 \times 10^3 \text{ tf cm}^{-2}, \quad \nu = 0.3, \quad p^1 = 1 \text{ tf cm}^{-1},$$

and noticing that p^1 takes the following values:

$$\overset{(m)}{\tilde{p}}{}^1 = -p^1 \cos(m \cdot 180°) = (-1)^{m+1} \cdot p^1 \tag{118}$$

the numerical results are shown in Fig. 8. From the results, the following matters are stated.

(1) The maximum distortion of the transverse cross-section, $\tilde{V}_1$, is about 2.59 times as large as the maximum transverse deflection, U_1, of the axial line, and is about 1/761 of the diameter of the shell.

(2) A lowering of the bending stiffness by the distortion of the transverse cross-section is affected by only the displacement $\tilde{V}_1$ of the circumferential wave number $m = 2$. The transverse deflection, U_1, of the axial curve increases slightly by this lowering. However, since the rate of this increase is about 1.004 times as large as the Bernoulli–Euler beam, this lowering of the bending stiffness is negligible for practical uses.

(3) The values of $\tilde{V}_1$ and U_1 are inversely proportional to the thickness, t, of the shell.

Now, if the in-plane distortion at the top of the shell is constrained, the boundary conditions are given by

$$\overset{(m)}{\tilde{V}}_\xi = 0 \qquad \text{at} \quad x = l \tag{119}$$

instead of eqn (109). Hence, the displacements, $\overset{(m)}{\tilde{V}}_\xi$, and the coefficients b and c given by eqn (114) become

$$\overset{(m)}{\tilde{V}}_\xi = \frac{\overset{(m)}{\tilde{p}}{}^\xi}{G \overset{(m,m)}{I}{}^{\xi\xi}} \frac{l^2}{2} \bar{x}(1-\bar{x}) \tag{120}$$

$$b = -\frac{p^1}{E_0 I^{11}} \frac{3\overset{(2)}{\tilde{p}}{}^1 l}{4G\pi t R^2}, \qquad c = \frac{p^1}{E_0 I^{11}} \frac{3\overset{(2)}{\tilde{p}}{}^1}{4G\pi t R^2}. \tag{121}$$

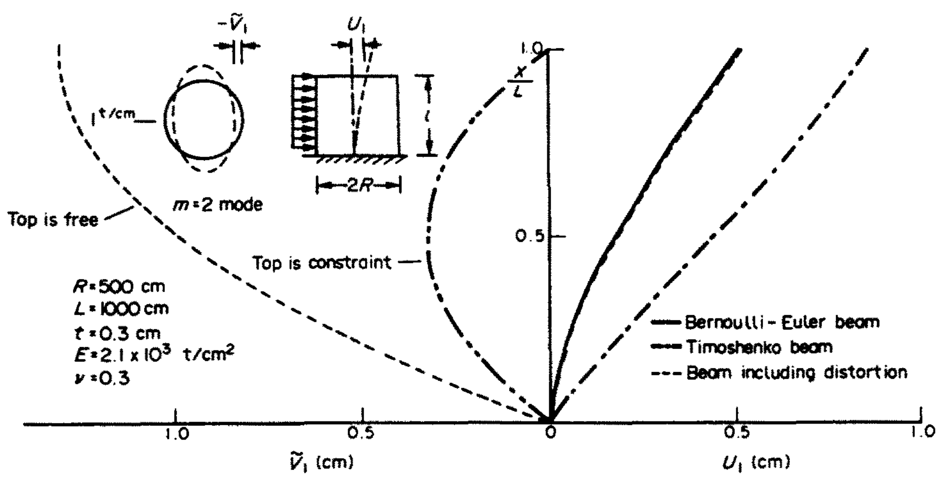

Fig. 8. Displacements U_1 and $\tilde{V}_1$ of cantilevered circular cylinders.

The numerical results of this case are shown in Fig. 8, and the following results are obtained.

(1) The maximum distortion in the case constrained the in-plane distortion at the top of shells is about 1/4 times as large as the maximum distortion in the unconstrained case.

(2) The maximum transverse deflection, U_1, and the maximum distortion are 1.0006 times and 0.65 times, respectively, as large as the maximum transverse deflection of the Bernoulli–Euler beam. Hence, it is effective in excluding the distortion to constrain the distortion at the top of the shell.

From the above the effect of the distortion of the transverse cross-section on the transverse displacement U_1 is negligible, we may use the Timoshenko beam hypothesis instead of the Bernoulli–Euler hypothesis used here for more accurate calculations.

9. CONCLUSIONS

The governing equations of motion for a circular cylindrical shell used in tanks and silos have been presented by rod theory including the distortion of the transverse cross-section. The governing equations are derived for infinitesimal deformations through the principle of virtual work, under the assumption that the deformation in the rod can be expressed by the higher-ordered deformation caused by the warping and distortion of the transverse cross-section added to the fundamental plane deformation expressed by the variations of the base vectors on the axial curve of the rod.

Also, using the classical hypotheses in rod theory, the simplifications of the governing equations have been discussed. Finally, the reliability of the derived theory has been shown by applying the Bernoulli–Euler beam theory, the Timoshenko beam theory, and the theory including the distortion to an elastic and static problem of a cantilevered circular cylindrical shell subjected to a uniform load. Also, for a circular cylindrical shell where the relation $t \ll R$ is valid, it has been shown that the influence of distortion on the constants of the cross-section is negligible. Therefore, the transverse deflection, U_1, of the axial line can be calculated approximately by the Bernoulli–Euler beam or more accurately by the Timoshenko beam, and this result is very useful in practice.

It is possible to extend the present theory to geometrically and materially non-linear problems, and the theory can be applied effectively further to buckling and dynamic problems of cylindrical shells.

Acknowledgements—The authors would like to express their gratitude to Dr Hiroaki Nagaoka of Nihon Kokan Inc. for his valuable discussions and careful reading of this manuscript.

REFERENCES

1. P. Seide, *Small Elastic Deformations of Thin Shells*, p. 430. Noordhoff, Leyden (1975).
2. Y. Hangai and H. Choi, Local buckling analysis of cantilevered cylindrical shells subjected to a transverse shearing force. *Trans. Archit. Inst. Japan* **346**, 71 (1984), in Japanese.
3. M. A. Haroun, Dynamic analysis of liquid storage tanks, EERL 80-04, Earthquake Engineering Research Laboratory, California Institute of Technology, Pasedena, California (1980).
4. G. W. Housner, Dynamic pressures on accelerated fluid containers. *Bull. Seism. Soc. Am.* **47**, 15 (1957).
5. P. M. Naghdi and M. B. Rubin, Constrained theories of rods. *J. Elasticity* **14**, 343 (1984).
6. W. Flügge, *Stresses in Shells*, 2nd Edn, p. 204. Springer, Berlin (1973).
7. G. R. Cowper, The shear coefficient in Timoshenko's beam theory. *J. Appl. Mech.* **33**, 335 (1966).
8. H. Takabatake and O. Matsuoka, The elastic theory of thin-walled open cross section with local deformations. *Int. J. Solids Structures* **19**, 1065 (1983).
9. H. Takabatake, The general theory of a thin-walled open section. *Trans. Archit. Inst. Japan* **279**, 13 (1978); **280**, 27 (1979), in Japanese.

APPENDIX: THE CONSTANTS OF THE CROSS-SECTION

If the effect of the distortion of the transverse cross-section is considered only for the geometrical moment of inertia, the constants of the cross-section defined by eqn (53) can be written approximately as follows for a uniformed circular cylindrical shell:

$$
\begin{aligned}
A &\simeq 2\pi t R \\
{}_0I^{\alpha\beta} &\simeq \pi t R^3\, \delta^{\alpha\beta}, \qquad \overset{*}{I}{}^{\alpha\beta}(n) = \tfrac{3}{2}\pi t R^2\, \hat{\delta}(n-2)\,(\delta_1^\alpha\delta_1^\beta - \delta_2^\alpha\delta_2^\beta) \\
\overset{(m,k)}{I}{}^{33} &\simeq \pi t R\, \hat{\delta}(m-k) \\
\overset{(m)}{I}{}_1^3 &\simeq 2m\pi t \qquad \text{(for odd } m\text{)} \\
\overset{(m)}{I}{}_\alpha^{\xi\beta} &\simeq \frac{\pi}{2} t R^2\, \hat{\delta}(m-2) \qquad (\text{for } \xi = \beta) \\
&\simeq -\frac{\pi}{2} t R^2\, \hat{\delta}(m-2) \qquad (\text{for } \xi \neq \beta) \\
\overset{(m,k)}{I}{}^{\xi\eta} &\simeq \pi t R \delta^{\xi\eta}\, \hat{\delta}(m-k) \\
\overset{(m)}{I}{}_1^{31} &\simeq 2m\pi t R \qquad \text{(for even } m\text{)} \\
\overset{(m,k)}{I}{}^{3\xi} &\simeq 2m\pi t \qquad \text{(for odd or even both } m \text{ and } k\text{).}
\end{aligned}
\tag{A1}
$$

All others are zero.

Where the notation $\hat{\delta}(m-k)$ is defined as follows:

$$
\begin{aligned}
\hat{\delta}(m-k) &= 1 \qquad (\text{for } m = k) \\
&= 0 \qquad (\text{for } m \neq k).
\end{aligned}
\tag{A2}
$$